SOME RESULTS ON THE USE OF P-SINGULAR FINITE ELEMENTS

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Resumen. El presente trabajo, resume, los resultados obtenidos mediante el uso de elementos finitos singulares y de transición de alto orden que modelizan la singularidad $r^{-1/2}$, aplicados a la mecánica de la fractura elástica lineal, a través del cálculo del factor de intensificación de tensiones, en un caso de fractura en modo I. Se muestra en primer lugar el la obtención, por medio del método de correlación de desplazamientos, de las expresiones de K_I para los diferentes elementos p-singulares utilizados. Se muestra como estos elementos pueden utilizarse de modo similar a como se venían utilizando otros elementos singulares hasta ahora, pero empleando un menor número de nodos con exactitud igual e incluso superior.

Abstract. This work contains the results obtained with p-singular finite elements in the case of linear elastic fracture. The stress intensification factors are calculated in the case of mode I fracture. We show how we can calculate the S.I.F. K_I by using the correlation displacements method in all the p-singular finite elements used. All it can be used similarly to others singular elements but increasing the accuracy of the results with a model less refinate.

1. INTRODUCTION

It is well know that the presence of a crack in a linear elastic solid induces singular strain fields near the crack tip. For a two-dimensional linear elastic fracture problem in terms of polar coordinates (r,θ) local to the crack tip, the near tip strain have the $r^{-1/2}$ form. This can be used in order to generate singular finite elements [2-5].

The singular elements of Barsoum [6] and Henshell and Shaw [7] have been used in elastic fracture problem with a mesh of quadratic finite elements. The singular elements of Pu, Hussain and Lorensen [8] have been used in elastic fracture problems with a mesh of cubic finite elements. We can thus increase the order of the singular element whilst at the same time increase the order of the order finite elements of the mesh. According to Babuska and Dorr [9] the finite element method does converge when we increase the order of the polynomial used even if we have singularities. The local error also decrease when one moves away from the singular point, Schatz and Wahlbin [10].

It is of interest to develop singular elements which are compatible with standard linear triangular and bilinear quadrilateral finite elements, and which at the same time have the possibility of allowing higher order polynimials to be used. We define some new singular finite elements having these special properties. It is shown that the strain approach in the singular element is appropiate [15].

For any problem containing a singularity ($r^{-1/2}$) the accuracy of the calculated solution will depend on the size of the singular elements which surround the singularity. According to Harrop[12] if the size of a singular element is reduced, the error in that element in representing the gradient is also reduced. However, in this case the region in which the singular gradient is represented effectively is itself reduced. This has motivated the use of so-called transition elements of the type proposed by Lynn and Ingraffea [13]. It is shown that the strain approach in the transition elements [15] (with some geometrical restrictions), is appropiate to the singularity involved.

2. STRESS INTENSIFICATION FACTORS

The presence of cracks in a material generally reduces the static strength of the material because the streses and strain are highly magnified at the crack tip. It has been established that parameters deduced from linear elastic fracture mechanics can be used to determine the stress and strain magnification at the crack tip. These parameters, stress intensity factor (SIF), incorporate applied stress levels, geometry and crack size in a systematic manner and may be evaluated from the elastic stress analysis of craked materials.

The stress intensity factors are computed using the displacements correlation technique.
Considering a crack problem in an infinite domain the displacements relative for the traction-free crack may be written in terms of an infinite series with respect to the polar coordinates r and θ.

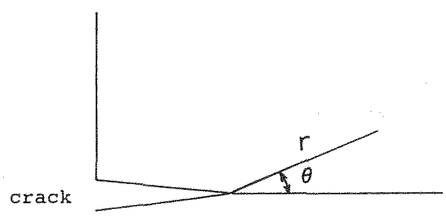

Figure 1

$$U = \frac{K_I}{4\mu}\sqrt{\frac{r}{2\pi}}\left[(2\kappa - 1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + \cdots$$
$$V = \frac{K_I}{4\mu}\sqrt{\frac{r}{2\pi}}\left[(2\kappa + 1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right] - \cdots \quad (1)$$

in wich

$$\mu = \textit{shear modulus} = \frac{E}{[2(1+\nu)]}$$

E = *Young's modulus*; ν = *Poisson's ratio*

$\kappa = (3-4\nu)$ *for plane strain;*

$\kappa = \frac{(3-\nu)}{(1-\nu)}$ *for plane stress.*

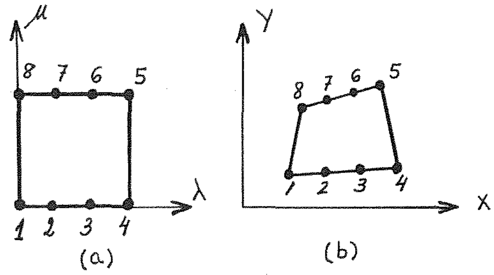

Figure 2

The general 2x4 Lagrange element in (x,y) space is as shown in fig.2(b). This is the image of the standar eight-node square element, $0\le\lambda,\mu\le 2$, in local (λ,μ) space under the transformation

$$x(\lambda,\mu) = \sum_{k=1}^{8} N_k(\lambda,\mu)\,x_k$$
$$y(\lambda,\mu) = \sum_{k=1}^{8} N_k(\lambda,\mu)\,y_k \quad (2)$$

where (x_k, y_k) are nodal points and N_k, (k=1,...8) the basis functions.

$$
\begin{aligned}
N_1 &= \frac{9}{32}\lambda^3\mu - \frac{9}{16}\lambda^3 - \frac{9}{8}\lambda^2\mu + \frac{9}{4}\lambda^2 + \frac{11}{8}\lambda\mu - \frac{11}{4}\lambda - \frac{\mu}{2} + 1\\
N_2 &= -\frac{27}{32}\lambda^3\mu + \frac{27}{16}\lambda^3 + \frac{45}{16}\lambda^2\mu - \frac{45}{8}\lambda^2 - \frac{9}{4}\lambda\mu + \frac{9}{2}\lambda\\
N_3 &= \frac{27}{32}\lambda^3\mu - \frac{27}{16}\lambda^3 - \frac{9}{4}\lambda^2\mu + \frac{9}{2}\lambda^2 + \frac{9}{8}\lambda\mu - \frac{9}{4}\lambda\\
N_4 &= -\frac{9}{32}\lambda^3\mu + \frac{9}{16}\lambda^3 + \frac{9}{16}\lambda^2\mu - \frac{9}{8}\lambda^2 - \frac{1}{4}\lambda\mu + \frac{1}{2}\lambda\\
N_5 &= \frac{9}{32}\lambda^3\mu - \frac{9}{16}\lambda^2\mu + \frac{1}{2}\lambda\mu\\
N_6 &= -\frac{27}{32}\lambda^3\mu + \frac{9}{4}\lambda^2\mu - \frac{9}{8}\lambda\mu\\
N_7 &= \frac{27}{32}\lambda^3\mu - \frac{45}{16}\lambda^2\mu + \frac{9}{4}\lambda\mu\\
N_8 &= -\frac{9}{32}\lambda^3\mu + \frac{9}{8}\lambda^2\mu - \frac{11}{8}\lambda\mu + \frac{1}{2}\mu
\end{aligned}
\quad (3)
$$

The desplacement components in the standard element are approximated by:

$$U(\lambda,\mu) = \sum_{k=1}^{8} N_k(\lambda,\mu)\,U_k$$
$$V(\lambda,\mu) = \sum_{k=1}^{8} N_k(\lambda,\mu)\,V_k \quad (4)$$

where (U_k, V_k) are nodal values. Rearranging (2) and (4)we have that:

$$
\begin{aligned}
x(\lambda,\mu) &= \alpha_1+\alpha_2\lambda+\alpha_3\mu+\alpha_4\lambda^2+\alpha_5\lambda\mu+\alpha_6\lambda^3+\alpha_7\lambda^2\mu+\alpha_8\lambda^3\mu\\
y(\lambda,\mu) &= \gamma_1+\gamma_2\lambda+\gamma_3\mu+\gamma_4\lambda^2+\gamma_5\lambda\mu+\gamma_6\lambda^3+\gamma_7\lambda^2\mu+\gamma_8\lambda^3\mu\\
U(\lambda,\nu) &= \beta_1+\beta_2\lambda+\beta_3\mu+\beta_4\lambda^2+\beta_5\lambda\mu+\beta_6\lambda^3+\beta_7\lambda^2\mu+\beta_8\lambda^3\mu\\
V(\lambda,\mu) &= \delta_1+\delta_2\lambda+\delta_3\mu+\delta_4\lambda^2+\delta_5\lambda\mu+\delta_6\lambda^3+\delta_7\lambda^2\mu+\delta_8\lambda^3\mu
\end{aligned}
\quad (5)
$$

where $\alpha_k, \beta_k, \gamma_k, \delta_k$, k=1....8, can be easily obtained.

If we apply all it to the side corresponding to μ=0, see figure 2(a) and taking account the mapping functions (3), we obtain for the vertical displacement V:

$$V = \left(-\frac{9}{16}\lambda^3+\frac{9}{4}\lambda^2-\frac{11}{\lambda}+1\right)\cdot V_1 + \left(\frac{27}{16}\lambda^3-\frac{45}{8}\lambda^2+\frac{9}{2}\lambda\right)\cdot V_2 + \left(-\frac{27}{16}\lambda^3+\frac{9}{2}\lambda^2-\frac{9}{4}\lambda\right)\cdot V_3 + \left(\frac{9}{16}\lambda^3-\frac{9}{8}\lambda^2+\frac{1}{2}\lambda\right)\cdot V_4 \quad (6)$$

being V_1, V_2, V_3 and V_4 the vertical displacements at nodes 1,2,3 and 4.
We can obtain a similar expresion for U_1, U_2, U_3 and U4.
And considering (2) for μ=0, we have

$$x = \left(-\frac{9}{16}\lambda^3+\frac{9}{4}\lambda^2-\frac{11}{44}\lambda+1\right).x_1 + \left(\frac{27}{16}\lambda^3-\frac{45}{8}\lambda^2+\frac{9}{2}\lambda\right).x_2 + \left(-\frac{27}{16}\lambda^3+\frac{9}{2}\lambda^2-\frac{9}{4}\lambda\right).x_3 + \left(\frac{9}{16}\lambda^3-\frac{9}{8}\lambda^2+\frac{1}{2}\lambda\right).x_4 \quad (7)$$

The geometrical restrictions that are neccesary in order to obtain the appropiate singularity are:

$$x_1 = 0, \quad x_2 = \frac{1}{9}l, \quad x_3 = \frac{4}{9}l \quad x_4 = l$$

l lenght of the element.
Making the subtitution in (7) we have

$$x = \frac{1}{4}\lambda^2 l \quad \textit{and then} \quad \lambda = 2\sqrt{\frac{r}{l}}$$

and taking account of (6)

$$V = V_1 + \frac{x}{l}\sqrt{\frac{r}{l}}\left(-\frac{9}{2}V_1+\frac{27}{2}V_2+\frac{27}{2}V_3+\frac{9}{2}V_4\right) + \frac{x}{l}\left(9V_1-\frac{45}{2}V_2+18V_3-\frac{9}{4}V_4\right) + \sqrt{\frac{x}{l}}\left(-\frac{11}{2_1}+9V_2-\frac{9}{2}V_3+V_4\right) \quad (8)$$

For the case of mode I fracture, see figure 3
$x = r$, $V_1 = 0$ (A), $V_2 = V_B$, $V_3 = V_C$, $V_4 = V_D$

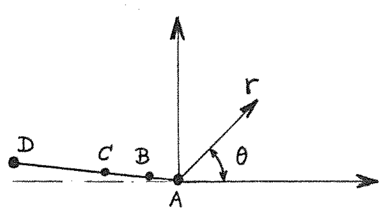

Figure 3

$$V = \frac{r}{l}\sqrt{\frac{r}{l}}\left(\frac{27}{2}V_B-\frac{27}{2}V_C+\frac{9}{2}V_D\right) + \frac{r}{l}\left(-\frac{45}{2}V_B+18V_C-\frac{9}{4}V_D\right) + \sqrt{\frac{r}{l}}\left(9V_B-\frac{9}{2}V_C+V_D\right) \quad (9)$$

being V_B, V_C and V_D the displacements ortogonal to the crack.
Considering the crack lenght along θ=180°, we have for (1)

$$V = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}(\kappa+1) \quad (10)$$

and if we compare (9) and (10) for the terms containing $\sqrt{r}$,

$$(9V_B-4.5V_C+V_D)\sqrt{\frac{r}{l}} = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}(\kappa+1)$$

and then

$$K_I = 2\sqrt{\frac{2\pi}{l}}\mu\frac{9V_B-4.5V_C+V_D}{\kappa+1} \quad (11)$$

Following the same analysis we obtain K_I for Lagrange elements of the 2x3 and 2x5 types.
Lagrange element type 2x3:

$$K_I = 2\sqrt{\frac{2\pi}{l}}\mu\frac{4V_B-V_C}{\kappa+1}$$

Lagrange element type 2x5:

$$K_I = 2\sqrt{\frac{2\pi}{l}}\mu\frac{16V_B-12V_C+\frac{16}{3}V_D-V_E}{\kappa+1}$$

3. NUMERICAL RESULTS

We considerer a case that has been studied by Hussain, vasilakis and Pu [14] and by Lynn and Ingraffea [13], corresponding to a double-edge-craked plate plane stress, figure 4.

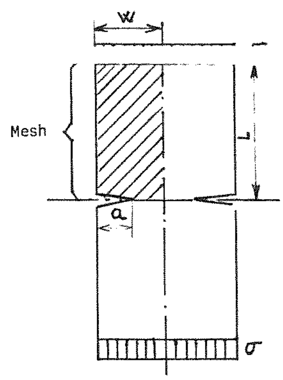

Figure 4

The plate of figure 4 is homogeneus with E = 5250 and ν=0.2. The thickness = 1.
The plate is subjected to a uniform tension of magnitude σ=1.

In this model (see figure 5) we employ first only singular elements with different numbers (2x3, 2x4 and 2x5) of nodes. We also employ the singular Serendipity elements of Barsoum [6] and Henshell and Shaw [7]. The calculations have been made for different values of the ratio a/c, and different integration order in the singular elements (See table 1). The accepted value of the stress intensity factor taken as a reference for this analysis was 2.815 (See Tada, Paris and Irwin [15]).

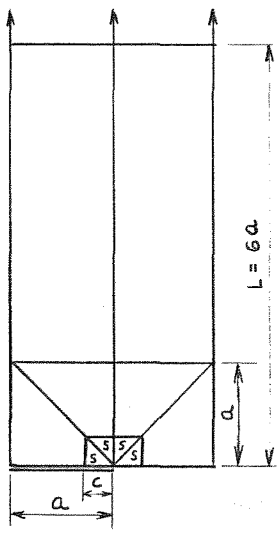

Figure 5

Mesh	P-singular Elements		Standard Finite Elements		Number of nodes of the Mesh
	Type	Integration order	Type	Integration order	
A	2 x 5	2 x 4 and 2 x 3	2 x 2	2 x 2	29
B	2 x 4	2 x 3 and 2 x 2	2 x 2	2 x 2	24
C	2 x 3	2 x 3 and 2 x 2	2 x 2	2 x 2	19
D	Serendipity 8 nodes	3 x 3 and 2 x 2	Serendipity 8 nodes	3 x 3 and 2 x 2	37

Table 1

These results are given in figures 6,7 and 8.

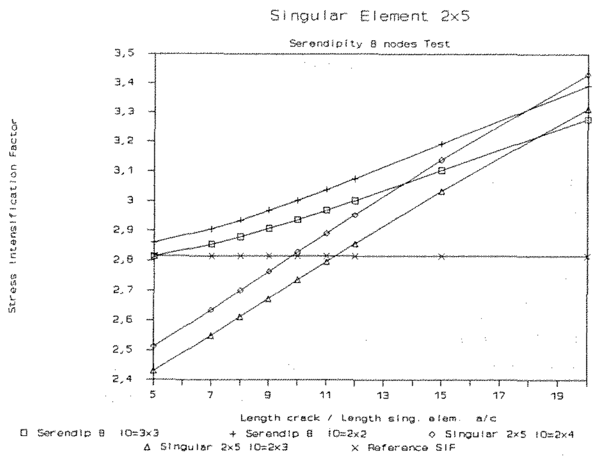

Figure 6

We can see the accuracy of the new approach as compared with the singular (Serendipity) elements.

The new approach has a bigger slope as compared with the singular (Serendipity) elements.

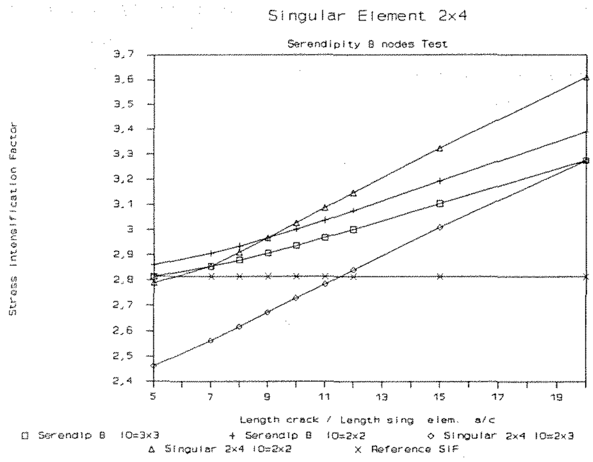

Figure 7

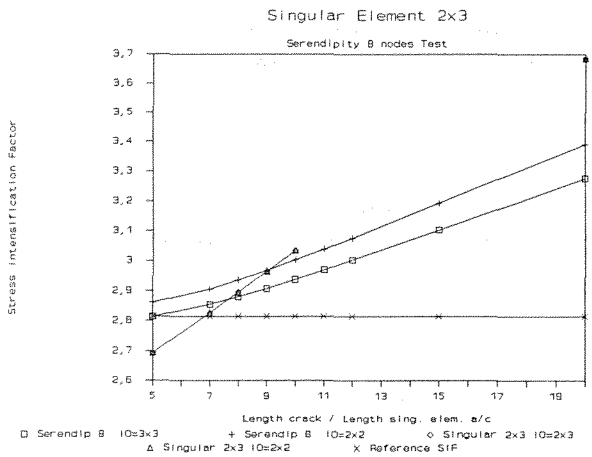

Figure 8

We considerer now, the same problem of figure 4 with 14 elements and introducing p-transition finite elements.

In the figure 9 we can see the mesh used with p-singular (S), p-transition (T) and standard finite elements.

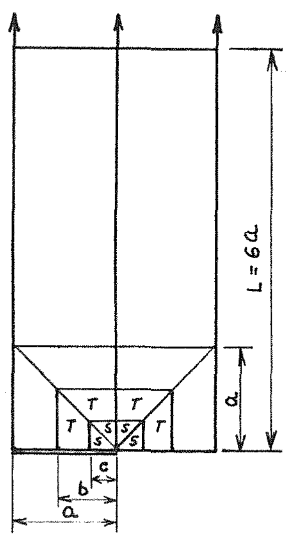

Figure 9

The p-singular (S) elements are 2x5, p-transition elements (T) are 2x3 and the standard elements are bilinear finite elements 2x2.

We can compare with other mesh using S,T, and N as Serendipity 8 nodes finite elements and with S and T appropiate (moving the intermediate nodes of some sides in order to incorporate the appropiate singularity).

In the following table 2 we can see the information about the elements we did use in the models.

Mesh	P-singular Elements		Transition Elements		Number of nodes of the Mesh
	Type	Integration order	Type	Integration order	
E	2 x 5	2 x 4 and 2 x 3	2 x 3	2 x 2	39
F	Serendipity 8 nodes	3 x 3 and 2 x 2	Serendipity 8 nodes	3 x 3 and 2 x 2	51

Table 2

In standard elements type 2x2 the integration order is 2x2 and in standard Serendipity 8 nodes finite elements are 3x3 and 2x2.

For all the cases we considerer the ratios a/c (a = length of the crack, c = length of the singular elements) and b/c (b = total length of the elements incorporating the singularity).

The results obtained for fixed a = 1.8 are show in figures 10, 11 and 12.

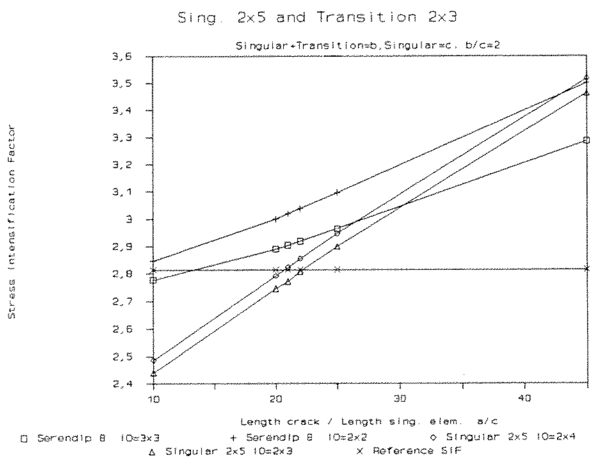

Figure 10

For a given ratio b/c we need decrease the length of the singular elements in order to obtain better results. Also it is shown that it is not neccesary to increase the integration order in order to obtain the best results.

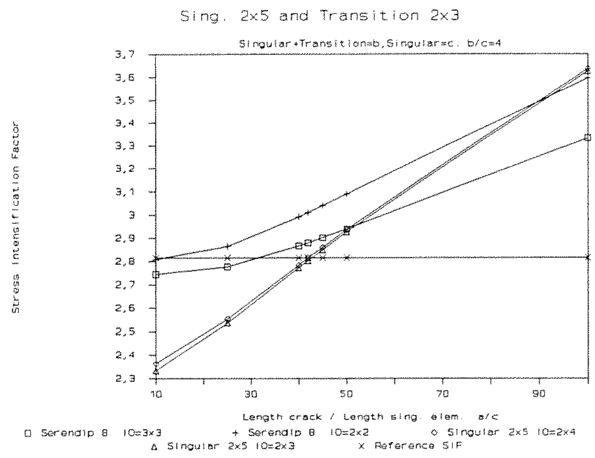

Figure 11

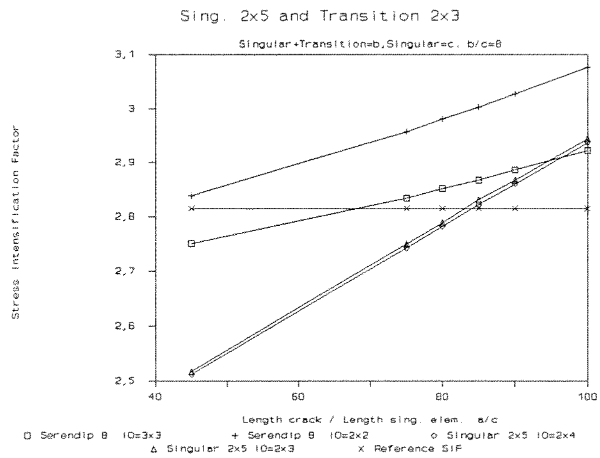

Figure 12

4. CONCLUSIONS

The 2xp, p≥3, p∈N, singular ($r^{-1/2}$) elements can be used to solve linear elastic fracture problems, exactly as for other singular elements, but with a smaller number of nodes and greater accuracy. This will become more important for three dimensional problems.

The principal advantage of the 2xp, p≥3, p∈N, singular ($r^{-1/2}$) elements is the possibility of increasing p in the radial direction and at the same time employing bilinear elements in the rest of the mesh. Obviously the same idea can be extended to 3xp, p≥3, p∈N singular ($r^{-1/2}$) elements in association with quadratic standard elements in the remainder of the mesh.

5. ACKNOWLEDGEMENTS

This Work was suported by the Consejería de Industria y Energia del Gobierno Autónomo de Canarias. This support is gratefull acknowledged.

6. REFERENCES

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