A NEW APPROACH TO THE TREATMENT OF SINGULARITIES WITH APPLICATION TO 3D CRACK PROBLEMS.

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Resumen. Cuando tratamos mediante técnicas de Análisis Numérico un problema que presenta un comportamiento singular, es muy ventajoso tener en cuenta la propia forma de la singularidad, si ésta es conocida. En esta comunicación se hace una generalización de la técnica de transformación cuadrática de Aalto al tratamiento de singularidades en un problema de elasticidad en tres dimensiones. Se desarrollan los elementos necesarios para poder tratar un caso de línea singular en tres dimensiones y se comparan los resultados con el caso correspondiente a una placa con una grieta circunferencial. Mediante esta técnica se pueden resolver problemas complejos de mecánica de fractura con mallados muy simples de elementos finitos. Se demuestra que el elemento finito singular al que se llega, reproduce correctamente la forma del campo de deformaciones singular en tres dimensiones en el caso de línea singular curva.

Abstract.- In any adaptation of a numerical technique for treating a singular situation, it is advantageous to have the knowledge of the form of the singularity. In this paper we extend the quadratic mapping technique to the treatment of singularities in three dimensional problems. Element types are developed for a three dimensional situation and comparison of results is made for a plate with circumferential crack. With the quadratic mapping technique we can perform fracture mechanics computations with rather simple finite element meshes. The element is shown to be able to reproduce a singular strain field in three dimensions for a curved singularity line.

1. INTRODUCTION

The analysis of the finite element method usually relies on the assumption that the solution of the given problem is regular enough. However, the implementation of the method is very often done on problems with polygonal domains which prevent the solution from being smooth in some points or lines. According Grisvard [1] the presence of corners lead to the singular behaviour of the solution only near the corners. This singular behaviour occurs even when the data of the problem are very smooth. It strongly affects the accuracy of the finite element method troughout the whole domain. A considerable body of analysis now exists showing that singularities can occur at such boundary points and lines,with the effect that the regularity of the solution is reduced from what is expected for such problems when the regions have smooth boundaries. However, many problems in potential theory and linear elasticity occur in regions which contain sharp corners and edges.

In any adaptation of a numerical technique for treating a singular situation, it is advantageous to have the knowledge of the form of the singularity. The form of the singularity is determined by the combination of geometry and boundary conditions. If we know the form of a singularity in general non convex domain in two dimensions, we can use the technique of Aalto [2] for locating the nodal points of standard isoparametric quadrilateral elements, properly around the singularity. The use of this quadratic mapped element for treating singularities has been shown by Michavila, Gavete, Díez and Whiteman [3,4,5]. Also, it has been demonstrated by Gavete, Michavila and Díez [6,7] that the Serendipity and Lagrange quadratic mapped elements can be applied in linear elastic fracture, because their strains and displacements in the vicinity of the crack tip, are appropiate to the form of the singularity.

In this paper we extend the quadratic mapping technique to the three dimensional problems. Element types are developed for a three dimensional situation and comparison of the results is made for a plate with circumferential crack. With the quadratic mapping technique we can perform fracture mechanics computations with rather simple finite element meshes. The element is shown to be able to reproduce a singular strain field in three dimensions for a circumferential singularity line. Other forms of singularity lines could be possible.

2.A NEW APPROACH TO THE TREATMENT OF 3-D CRACK PROBLEMS.

A new formulation for the treatment of singularities in seepage problems was established by Aalto [2]. It is based on the analytical solution of the problem in an infinite medium and gives simple formulae for locating the nodal points of standard quadrilateral elements properly around the singular point. The mapping is as follows:

$$x\ (\lambda,\mu) = \frac{a}{4}(\lambda^2-\mu^2)$$
$$y\ (\lambda,\mu) = \frac{a}{2}\lambda\,\mu \qquad (1)$$

being $a > 0$ and $\lambda, \mu \in [0,2]$.

The differences and advantages of the use of this mapping over the quarter point elements are:

(1) We need less singular elements to model the singularity.
(2) These singular elements can be used not only when the singular solution for strains near the crack tip is of the form $r^{-1/2}$, but in more general problems (with the appropiate mapping instead of (1)).

In the three-dimensional context our interest is in the finite element representation of line singularities, so that strain forms in the elements in ortogonal planes to the lines of singularity are considered.

The 20-node isoparametric brick element in physical (x,y,z) space (Fig. 1a) is obtained from the standard 20-node element (Fig. 1b) in local (λ,μ,ν) space by mapping defined in (2) as follows, for this case

$$x = \frac{a}{4}(\lambda^2 - u^2)$$
$$y = \frac{a}{2}\lambda\,\mu \qquad (2)$$
$$z = b\,\nu$$

being $\lambda, \mu, \nu\ \varepsilon\ [0,2]$

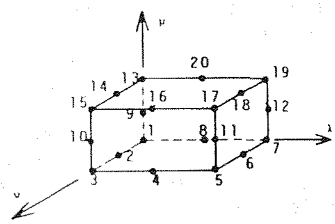

Fig.1a.

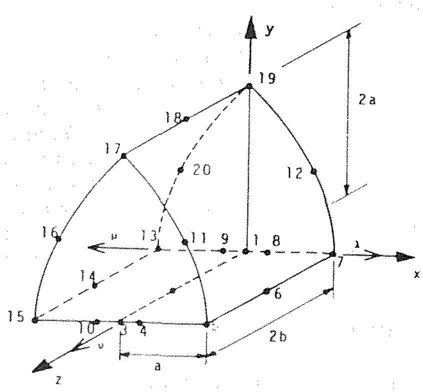

Fig.1b.

However we are interested in the case of a circunferential crack problem. So we need to develop a different and more complicate mapping.

In order to do it, our starting point is the work of Aalto, Gavete, Michavila and Díez [2-7], and in this paper we calculate a new mapping, giving a special singular finite element, which is used to approach a 3-D circumferential crack problem. The strain form approach for this new singular element is appropiate to the singularities involved, as it is demonstrated in this paper. The mapping is as follows:

$$x = \frac{a}{4}(\lambda^2-\mu^2)+\left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right](1-\cos\frac{v\alpha}{2})$$

$$y = \frac{a}{2}\lambda\mu \qquad (3)$$

$$z= \left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right] \operatorname{sen}\frac{v\,\alpha}{2}$$

being $\lambda, \mu, v \in [0, 2]$

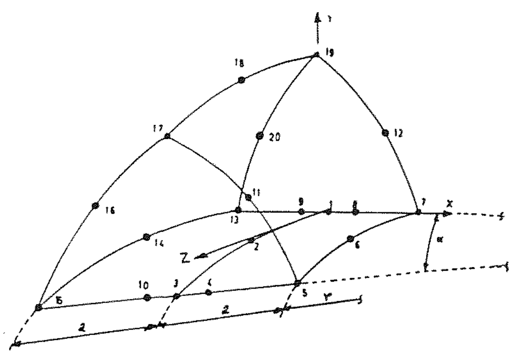

Fig.2.

This mapping is applied to the standard 20-nodes brick finite element giving the element of the fig. 2 in physical space. The singularity line (crack line) is placed along 1 - 2 - 3.

The approximations of the strains ε_{11}, ε_{22} and ε_{12} in planes ortogonal to the crack line are

$$E_{11}=\frac{\partial U(\lambda,\mu,v)}{\partial x}=\frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x}+\frac{\partial U}{\partial \mu}\frac{\partial \mu}{\partial x}+\frac{\partial U}{\partial v}\frac{\partial v}{\partial x}$$

$$E_{22}=\frac{\partial V(\lambda,\mu,v)}{\partial y}=\frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial y}+\frac{\partial V}{\partial \mu}\frac{\partial \mu}{\partial y}+\frac{\partial V}{\partial \mu}\frac{\partial v}{\partial y} \qquad (4)$$

$$E_{12}=\frac{1}{2}\left[\frac{\partial U(\lambda,\mu,v)}{\partial y}+\frac{\partial V(\lambda,\mu,v)}{\partial x}\right]$$

For the mapping (3) we have

$$|J| \equiv \left|\frac{\partial(x\,y\,z)}{\partial(\lambda\mu v)}\right| \equiv \frac{\alpha\,a^2}{8}\left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right](\lambda^2+\mu^2) \qquad (5)$$

The forms of U (λ, μ, v), V(λ, μ, v) are

$$U(\lambda,\mu,v) = \alpha_1+\alpha_2\lambda+\alpha_3\mu+\alpha_4 v+\alpha_5\lambda^2+\alpha_6\mu^2+\alpha_7 v^2$$
$$+\alpha_8\lambda\mu+\alpha_9\lambda v+\alpha_{10}\mu v+\alpha_{11}\lambda^2\mu+\alpha_{12}\lambda\mu^2+\alpha_{13}\mu^2 v+\alpha_{14}\mu v^2+$$
$$+\alpha_{15}\lambda^2 v+\alpha_{16}\lambda v^2+\alpha_{17}\lambda\mu v+\alpha_{18}\lambda^2\mu\, v+\alpha_{19}\lambda\mu^2 v+\alpha_{20}\lambda\mu v^2 \qquad (6)$$

and similarly for V (λ, μ, v) changing, α_i by β_i (i=1, ...20)

$$E_{11}=\left[\frac{\partial U}{\partial \lambda}\frac{a\lambda\alpha}{4}\cos\frac{v\,\alpha}{2}\left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right]\right]$$
$$+\frac{\partial U}{\partial \mu}\frac{a\alpha\mu}{4}\left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right]$$
$$-\frac{\partial U}{\partial v}\left[(r+a) - \frac{a}{4}(\lambda^2-\mu^2)\right]$$
$$\left[\frac{a\alpha\lambda}{4}\operatorname{sen}\frac{v\,\alpha}{2}\right] \;/\; |J| \qquad (7)$$

and for $\lambda = 0$

$$E_{11}=\frac{\alpha_3+2\alpha_6\mu+\alpha_{10}v+2\alpha_{13}\mu v+\alpha_{14}v^2}{\frac{a}{2}\mu} \qquad (8)$$

and hence for v = cte, i.e. in a plane ortogonal to the crack line

$$E_{11}=\tilde{A}_1+\frac{\tilde{A}_2}{\mu}= A_1+\frac{A_2}{\sqrt{r}} \qquad (9)$$

where, $\tilde{A}_i$ and A_i (i=1,2) are constants.

Similar results we obtain for ($\mu = 0$, v = constant) and for (λ=Kμ, v=constant), radial lines emanating from the singularity in a plane ortogonal to the crack line (1 - 2 - 3 of fig. 1), where K > 0 is a constant. Also we obtain similar results for E_{22} and E_{12} (v = constant). Thus for small r the $r^{-1/2}$ term dominates in all directions emanating from the line of singularity in planes ortogonal to this line, giving the approximations to the gradients the correct "$r^{-1/2}$" singular form as required by the true solution. Similar results to those obtained in the case of mapping the 20-nodes brick element, can be obtained by mapping the 27-nodes quadratic isoparametric Lagrange brick element in a similar way.

3. AN EXAMPLE CONTAINING A SINGULARITY LINE.

The 3-Dimension crack models represent a very important problem nowadays in mechanical engineering, considering that pipe and pressure vessel cracks can be detected, before the broken pressure arrives. Width and length intervene very directly in this models, and even the crack shape is important.

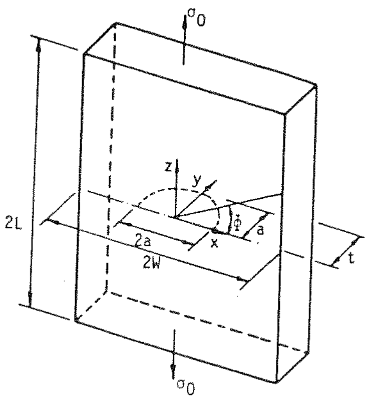

Fig. 3

As an example we have studied the problem of a plate with a circumferential crack which is under an uniform stress field.

Figure 3 shows plate, subject to an uniform stress field and having a circular center crack. Data are a/t= 0.4, L/t= 2.5 and a/w= 0.2.

In order to use the new mapping defined in (3) for the case of circumferential crack line, fig.4 model, has been carried out. After the displacements have been obtained we use the Manu's formula [8], to calculate the SIF.

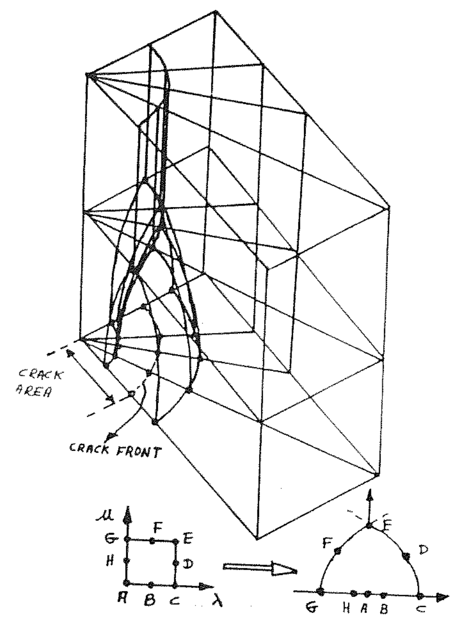

Fig. 4

As a difference with the quarter point elements, here we don't have negative pivots during the factorization which is carried out in the equations system solution or other numerical difficulties as reported by Peano and Pasini in [9]. The results obtained for S.I.F. versus the orientation (Φ angle of figure 2) are shown in figure 5.

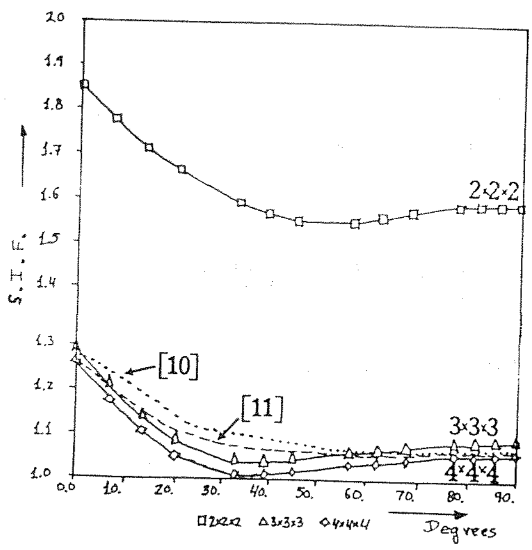

Fig.5

The performed errors when compared with references [10] and [11] can reach up to 5% and even lower, which represents an acceptable error level, if we consider that reference models of Miyazaki, Watanabe and Yagawa [10] and Yagawa, Ichimiya and Ando [11] have more finite elements. In general, these errors depend on the numerical integration order, on the diameter and on the elements aspect ratio of the model.

Also a very good agreement has been obtained with the accurate results of Schnack and Karaosmanoglu [12].

4. CONCLUSIONS.

We have compared ours curves of the SIF values versus the angle Ø, with those obtained by Miyazaki, Watanabe, Ichimiya, Ando and Yagawa [10-11], when they solve a similar example with different strategies. Their results are very much the same than ours (integration orders 3 and 4), but our grid is much less refinate. This method has the advantage of its simplicity, it is possible to solve problems with much less refinate grids. We have demonstrated that a new singular element can approach the behaviour of curved cracks in 3-D, calculating with high precission the stress intensity factor. Others forms of singularity lines could be possible. It would be convenient also to study the influence of the geometric shape of the crack.

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