# ELASTOPLASTIC CONTACT IN AN INDENTATION IMPACT TEST BY FRACTIONAL CALCULUS

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## ABSTRACT

This communication presents an elasto-viscoplastic contact model of an impact-indentation test by means of fractional calculus. The models aims at characterising this contact by a rheological model based on the superposition in series of a dashpot and a non linear spring. The dashpot gives the dissipative character and the superposition in series the permanent character of the strains. Finally, the spring gives the non linear stiffness based on Hertz theory.

The model treatment is achieved by fractional calculus, which is solved numerically using G1 method. Then, the results are compared with those obtained from experimental tests.

Finally, material properties have been obtained from the contact characterisation.

## RESUMEN

En esta comunicación se presenta un modelo de contacto elasto-viscoplástico de un ensayo de impacto-indentación mediante la aplicación del cálculo fraccionario. El modelo persigue caracterizar dicho contacto mediante un modelo reológico basado en la superposición en serie de un amortiguador y un muelle no lineal. El amortiguador proporciona el carácter disipativo y la superposición en serie el carácter permanente de las deformaciones. Finalmente, el muelle proporciona la rigidez no lineal basada en la teoría de Hertz.

El tratamiento del modelo se efectúa por medio del cálculo fraccionario que se resuelve numéricamente utilizando el método G1. Posteriormente, se realiza una comparación de los resultados con los obtenidos de ensayos experimentales.

Finalmente, a partir de la caracterización del contacto se han obtenido las propiedades del material.

**KEY WORDS:** Fractional calculus; Hertz contact, Impact-Indentation.

## 1. INTRODUCTION

The use of polymeric materials has been continuously increasing in the last years within the automotive industry not only to improve fuel efficiency and crashworthiness but also for their energy dissipation capacity, possibility of recycling, design versatility and good surface quality [1]. Besides, they can be reinforced so that they increase their specific mechanical properties.

In order to design the polymeric components (material, geometry, dimensions...) in the automotive industry, the material behaviour against impact is a key point. Nevertheless, material properties (characterisation and material model) become especially important, since they depend highly on the strain rate at which they are loaded [2].

In almost all the practical situations, the parts or components are subjected to complex loads which induce a complex response in the component where non homogeneous strain and strain rate distributions prevail. This complex response can be divided into two other ones: the structural global response, in which both the material and geometry take part, and the local response, in which the material response prevails.

To fully define the local response, the use of a rheological model, based on the superposition in series of a dashpot and a non linear spring, is proposed to solve the contact problem during impact-indentation, based on Hertz theory [3, 4].

Contact problems induce non-linear equations which generally can be solved after fulfilling an arduous and complex labour.

Hertz theory is restricted to non conforming surfaces, continuous and frictionless, and to perfectly elastic solids subjected to small deformations. Contact between non conforming surfaces takes place at a point or along a line and generally, in spite of the load, contact zone dimensions are small in comparison with the solids size. Under this simplification, a local concentration of stress is originated that may be analysed regardless of the global stress distribution resulting in the solids. [6]

This paper deals with the resolution of the non-linear equations derived from the contact problem during indentation following Hertz theory by means of transforming the governing non-linear equations into linear fractional integro-differential equations.

First, a brief theoretical background is presented concerning to fractional calculus, laying emphasis on the definitions of Günwald-Letnikov and numerical treatment. Next, the experimental technique used for the impact-indentation tests is described. Then the equations resulting from the rheological model are solved after having been transformed making use of fractional calculus, obtaining a linear integro-differential fractional equation that is numerically solved. Finally, the results obtained are compared with those provided by experimental tests.

## 2. THEORETICAL BACKGROUND

Fractional calculus is a discipline that has historically been relegated to the theoretical mathematics, but during last decades, different applications have been attributed to this branch of mathematical analysis. [7, 8, 9, 10].

Next, two definitions of fractional derivatives are described: the ones of Riemman-Liouville and Grünwald-Letnikov.

The definition of Riemman-Liouville is generally employed for non-integer order integrals. Indeed, the $\alpha$ order derivative with respect $t$ variable for a function $f(t)$, with $\alpha < 0$ and the lower integration limit being zero, is defined as

$$\mathrm{D}^{\alpha} f(t) = \frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-y)^{-\alpha-1} f(y)\,\mathrm{d}y \quad (1)$$

where $\Gamma(z)$ is the gamma function of real argument $z$, given by

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-y}\, y^{z-1}\,\mathrm{d}y\,. \quad (2)$$

This function represents the generalisation of the factorial function, satisfying

$$z\,! = \Gamma(z+1)\,, \quad (3)$$

which coincides with the classic definition of the factorial if $z$ is an entire number. For example, for $-1 < \alpha < 0$, the application of Eq. (1) on the function $t^q$ yields

$$\mathrm{D}^{\alpha} t^q = \frac{\mathrm{d}^{\alpha} t^q}{\mathrm{d}t^{\alpha}} = \frac{\Gamma(q+1)}{\Gamma(q+1-\alpha)}\, t^{q-\alpha}\,. \quad (4)$$

This result is employed in Section 3 to transform the non-linear equations characterising the contact mechanics into linear integro-differential fractional equations.

The Grünwald-Letnikov definition of the fractional derivatives, for any real order $\alpha$, arises from the backward definition of the n entire order derivative. Indeed, the first derivative is given by

$$\mathrm{D}^1 f(t) = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t}\,, \quad (5)$$

the second by

$$\mathrm{D}^2 f(t) = \lim_{\Delta t \to 0} \frac{f(t) - 2f(t-\Delta t) + f(t-2\Delta t)}{(\Delta t)^2}\,, \quad (6)$$

the third by

$$\mathrm{D}^2 f(t) = \lim_{\Delta t \to 0} \frac{f(t) - 3f(t-\Delta t) + 3f(t-2\Delta t) - f(t-3\Delta t)}{(\Delta t)^3}\,,$$

(7)

and so on, thus the $n$ order derivative satisfies

$$\mathrm{D}^n f(t) = \lim_{\Delta t \to 0} \left( (\Delta t)^{-n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} f(t - j\Delta t) \right), \quad (8)$$

where:

$$N = t \,/\, \Delta t\,, \quad (9)$$

and the Newton binomial

$$\binom{n}{j} = \frac{n\,!}{j\,!(n-j)!}\,, \quad (10)$$

have been employed. If the gamma function is applied, considering that

$$(-1)^j \binom{n}{j} = \binom{j-n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)}\,, \quad (11)$$

thus Eq. (8) may be transformed into

$$\mathrm{D}^n f(t) = \lim_{\Delta t \to 0} \left( (\Delta t)^{-n} \sum_{j=0}^{N-1} \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} f(t - j\Delta t) \right). \tag{12}$$

If in this equation the $n$ order is substituted by any real order $\alpha$, the Grünwald-Letnikov definition for fractional derivatives yields

$$\mathrm{D}^\alpha f(t) = \lim_{\Delta t \to 0} \left( (\Delta t)^{-\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} f(t - j\Delta t) \right) \tag{13}$$

or finally,

$$\mathrm{D}^\alpha f(t) = \lim_{\Delta t \to 0} \left( (\Delta t)^{-\alpha} \sum_{j=0}^{N-1} A_{j+1} f(t - j\Delta t) \right), \tag{14}$$

where the terms $A_{j+1}$ are the so-called Grünwald-Letnikov coefficients, satisfying

$$A_{j+1} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)}. \tag{15}$$

In Eq. (14) it should be remarked that the fractional derivative is constructed employing all the history of the function, weighted by the Grünwald-Letnikov coefficients, putting in evidence the memory of fractional operator. To avoid the use of the gamma function in numerical applications, the following properties of the weighting coefficients may be employed:

$$A_1 = 1, \tag{16}$$

$$A_{j+1} = \frac{j - \alpha - 1}{j} A_j, \tag{17}$$

and, if $\alpha < -1$,

$$\lim_{\Delta t \to 0} A_{j+1} = 0. \tag{18}$$

This last property is known as the fading memory of fractional derivatives, implying the most recent history is more influent than the fastest.

The fractional operator can be numerically calculated by [9]:

$$\mathrm{D}^\alpha f(t) = \left( \frac{N}{t} \right)^\alpha \sum_{j=-1}^{N} w_j(\alpha)\, f\left( t - j\frac{t}{N} \right), \tag{19}$$

which is analogue to the conventional quadrature formulae, where $w_j(\alpha)$ are the weighting coefficients, depending on the derivation order. To simplify the nomenclature, Eq. (19) may be also written as

$$\mathrm{D}^\alpha f(t) = \frac{1}{(\Delta t)^\alpha} \sum_{j=-1}^{N} w_j\, f_j\ . \tag{20}$$

One of the most employed methods is the G1 one, based on the definition of Grünwald-Letnikov.

The G1 method is generally more efficient for derivation orders comprised between 0 and 1. The weighting coefficients $w_j$ are given by

$$w_j(\alpha) = \begin{cases} 0 & \text{if } \ j = -1, N; \\ \dfrac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} & \text{if } \ 0 \le j \le N-1; \end{cases} \tag{21}$$

which may be also evaluated form the recurrence Eq. (17).

This numerical method will be employed in Section 4 to solve the Hertz contact problem enunciated later.

## 3. MATERIAL AND EXPERIMENTAL TECHNIQUE

The experimental part was presented in a previous work [10]. The material used is an injection grade isotactic Polypropylene (PP) homopolymer (SM6100K, Montell). Specimens are $4\ \mathrm{mm}$ thick and $80\ \mathrm{mm}$ diameter circular plates, and are subjected to instrumented-indentation-impact tests.

The tests are carried out in a Dartvis falling weight test machine (Ceast). A $0.7437\ \mathrm{kg}$ striker with a $12.7\ \mathrm{mm}$ diameter hemispherical dart is released from different heights. The striker hits the sample that lies on a rigid $10\ \mathrm{mm}$ thick steel surface, inducing a local indentation on the surface of the sample.

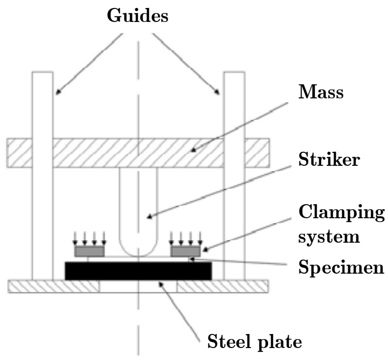



Fig. 1 Indentation-impact test configuration in a falling weight impact machine.

Up to 8 tests have been carried out in different sectors of each specimen, as the area affected by the contact strain is very small compared to the size of the whole specimen.

Tests have been carried out from $5\ \mathrm{mm}$ to $100\ \mathrm{mm}$ height, with increments of $5\ \mathrm{mm}$. Each test has been repeated 3 times to analyse reproducibility. The influence of the thickness of the specimen has also been analysed, by superposing two or three samples and comparing force-time curves.

As a result, force-time curves show a good reproducibility, even if the curves must be displaced in time to avoid the dynamical effect of the beginning of the curves i.e. due to specimen accommodation effects [5].

Experimental force-time curves show a quasi-symmetrical shape, increasing the maximum force and decreasing contact time as the impact energy increases [5].

## 4. RHEOLOGICAL MODEL

The response of polymeric materials subjected to impact-indentation loads may be described by the application of a rheological model based on Hertz contact theory [4, 5]. This model is based on the superposition in series of a dashpot and a non linear spring. The dashpot gives the dissipative character and the superposition in series the permanent character of the strains. Finally, the spring gives the non linear stiffness based on Hertz theory.

Figure 1. shows the rheological model of the impact-indentation test, in which $m$ represents the striker mass, $k$ the constant of the non linear spring and $c$ the constant of the dashpot.

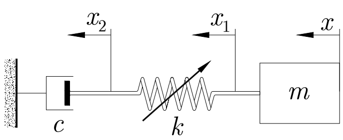



Fig. 2. Rheological model

The constant $k$ is related to the materials' properties as follows:

$$k = \frac{4\sqrt{R}}{3}\left(\frac{1-\nu_\mathrm{i}^2}{E_\mathrm{i}} + \frac{1-\nu_\mathrm{s}^2}{E_\mathrm{s}}\right)^{-1} \quad (22)$$

where:

$R$ is the radius of the striker, $E_\mathrm{i}$ and $E_\mathrm{s}$ the Young modulus of the striker and specimen, respectively; $\nu_\mathrm{i}$ and $\nu_\mathrm{s}$ the Poisson modulus of the striker and specimen, respectively [5].

Following the model of Fig. 2, the governing equations for an impact-indentation test with initial relative velocity $\dot{x}(0) = \dot{x}_0$, which depends on the falling height of the mass $h$, are given by:

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx_1^{3/2} = -c\frac{\mathrm{d}x_2}{\mathrm{d}t} \quad (23)$$

Taking into account the following expression, obtained from the model:

$$x = x_1 + x_2 \quad (24)$$

and eliminating $x_2$, Eq. (23) may be written as:

$$m\ddot{x} = -kx_1^{3/2} = -c\left(\dot{x} - \dot{x}_1\right) \quad (25)$$

where the operator $(\ddot{\bullet})$ and $(\dot{\bullet})$ denote second and first derivatives with respect to time, respectively. This non-lineal differential equation may be solved, for example, by means of Runge-Kutta family algorithms.

From the solution of the $\alpha$ order derivative for the function $t^q$ with respect to the variable $t$ indicated in Eq. (4), taking, $q = 1$, and $\alpha = -1/2$, the term $x_1^{3/2}$ may be related with the semi-integral $\mathrm{D}^{-1/2}x_1$ by

$$\mathrm{D}^{-1/2}x_1(t) = \frac{4}{3\sqrt{\pi}}x_1^{3/2}(t)\,, \quad (26)$$

where $\Gamma(2) = 1$ and $\Gamma(5/2) = 3\sqrt{\pi}/4$ have been taken into account.

Taking into account Eq. (26), Eq. (25) is transformed into the linear integro-differential equation

$$m\ddot{x} = -k\frac{3\sqrt{\pi}}{4}\mathrm{D}^{-1/2}x_1 = -c\left(\dot{x} - \dot{x}_1\right) \quad (27)$$

where it should be pointed out that operator $\mathrm{D}^{(\bullet)}$ represents fractional derivative with respect to the variable $x$ instead of $t$ [11].

Aimed at exploring the capabilities of the fractional calculus to solve problems relative to contact mechanics, the integro-differential equations (27) are numerically solved, making use of the G1 method. To do so, an algorithm based on the central finite difference is proposed [11]. For that, uniform increments for the time $\Delta t$ are considered. Hence, at the instant $t^n$, Eq. (27) becomes

$$m\ddot{x}^n = -k\frac{3\sqrt{\pi}}{4}\mathrm{D}^{-1/2}x_1^n = -c\left(\dot{x}^n - \dot{x}_1{}^n\right). \quad (28)$$

The fractional operator may be discretised by Eq. (20), giving

$$m\ddot{x}^n = -\frac{3k}{4}\sqrt{\pi\,\Delta x_1}\sum_{j=0}^{n} w_j x_1^{n-j} = -c(\dot{x} - \dot{x}_1)\,, \quad (29)$$

where, as in the previous section, it has been taken into account that G1 method has $w_{-1}$ coefficient equal to zero. The weighting coefficients $w_j$ are taken from Eqs. (21).

For the described case, the interval $\Delta x_1$ varies for each $t_n$ instant, satisfying

$$\Delta x_1 = \frac{x_1^n}{n}. \tag{30}$$

Indeed, any forward displacement would be given by

$$x_1^{n-j} = (n-j)\Delta x_1\,, \tag{31}$$

conducing to the following equations:

$$\begin{cases} m\ddot{x}^n = -\dfrac{3}{4}k\sqrt{\pi}\left(\dfrac{x_1^n}{n}\right)^{3/2}\displaystyle\sum_{j=0}^{n} w_j\,(n-j) \\ m\ddot{x}^n = -c(\dot{x}-\dot{x}_1) \end{cases}. \tag{32}$$

This lineal differential equation may be solved by means of central finite difference method [11]. The velocity $\dot{x}^n$ and acceleration $\ddot{x}^n$ are given by

$$\dot{x}_1^n = \frac{x_1^{n+1} - x_1^{n-1}}{2\Delta t} \tag{33}$$

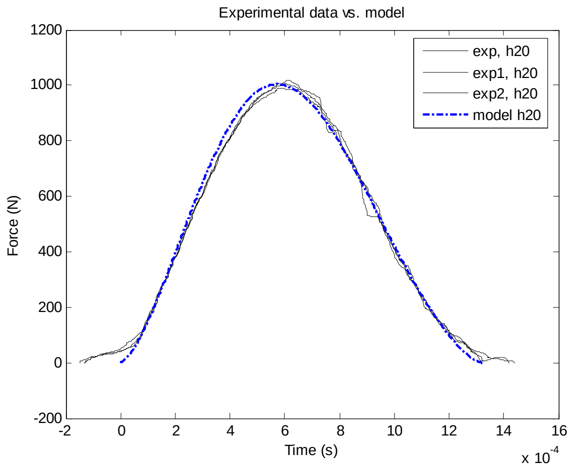
$$\ddot{x}^n = \frac{x^{n+1} - 2x^n + x^{n-1}}{(\Delta t)^2} \tag{34}$$

Consequently, Eqs. (32) lead finally to

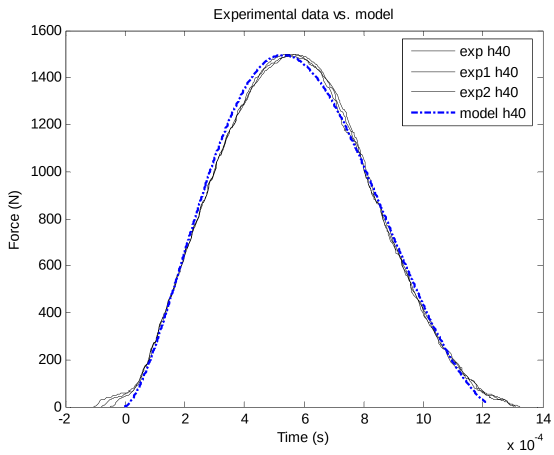
$$\begin{aligned} x^{n+1} &= 2x^n - x^{n-1} - (\Delta t)^2\frac{3k}{4m}\sqrt{\pi}\left(\frac{x_1^n}{n}\right)^{3/2}\sum_{j=0}^{n} w_j(n-j) \\ x_1^{n+1} &= x_1^{n-1} + 2\Delta t\left(\frac{m}{c}\ddot{x}^n + \dot{x}^n\right) \end{aligned} \tag{35}$$

After solving the equations, $k$, and therefore $E_s$, and $c$ may be calculated by fitting the results to the experimental ones [5]. To do so, maximum force $F_{max}$ and restitution coefficient $\varepsilon$ have been used.

Figure 3 represents the impact force for a drop height of a) $h = 20$ mm and b) $h = 40$ mm obtained in experimental tests and by the numerical model.




(a)




(b)

Fig. 3. Experimental impact force and the one computed by fractional calculus.

The values of $F_{max}$, $\varepsilon$, $k$, $c$, and $E_s$, found for 2 different falling heights, $h = 20$ mm (h20 test) and $h = 40$ mm (h40 test) are shown in Table 1.

Table 1. Parameters for the rheological model.

| Test | $F_{max}$ [N] | $\varepsilon$ | $k$ [kNm$^{-3/2}$] | $c$ [Nsm$^{-1}$] | $E_s$ [GPa] |
|---|---|---|---|---|---|
| h20 | 1004 | 0.59 | 329900 | 6080 | 2.64 |
| h40 | 1495 | 0.56 | 332000 | 6060 | 2.66 |

The elasticity moduli obtained are 2.64 GPa and 2.66 GPa for h20 and h40 tests, respectively. These values are more than twice the value obtained from impact-tensile tests (1,2 GPa) [5, 12]. Therefore, it is shown the deformation pattern dependence upon material properties.

## 5. CONCLUSIONS

In this communication a contact problem resulting from an impact-indentation test has been solved using fractional calculus. The proposed method consists in transforming the non-linear equation that governs the contact in a fractional integro-differential equation. The numerical solution has been carried out by means of G1 method, based on the fractional derivative definitions of Grünwald-Letnikov.

From the analysis of the results it may be stated that the elasticity modulus of PP has been obtained from the rheological model. The values obtained are more than twice the value obtained from impact-tensile tests.

Concluding, it has been proved that the fractional calculus is able to study certain mechanical contact problems. Thus more complex applications could be investigated in the future.

To better fit to experimental data, other rheological models may be used as well as new types of damping.

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